

Extremal Kähler metrics and energy functionals on projective bundles

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1 Introduction

In [5], E. Calabi introduced the extremal Kähler metric on a compact Kähler manifold, which is a critical point of the Calabi functional. The existence of extremal Kähler metrics is a long standing difficult problem, which is closely related to some stabilities conditions in algebraic geometry. In the special case of projective bundles, it is showed in literatures (cf. [5][15][18][3] etc.) that the extremal metrics can be explicitly constructed and have many interesting properties. However, there exists a Kähler manifold which admits no extremal metrics in certain Kähler classes. Thus, a natural question is whether there are extremal metrics with singularities on such manifolds and how the energy functionals behaves. In the

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present paper, using the construction of [3] we will study the relation between the existence of extremal Kähler metrics and energy functionals on projective bundles.

The extremal metrics with conical singularities are studied on Riemann surfaces in [7][21]. Similar to the smooth case, it is believed that the existence of conical extremal metrics is related to the behavior of energy functionals as well as some stability conditions, as discussed by Donaldson in [12][13]. Based on the construction of extremal metrics on projective bundles in [3], we have the result:

Theorem 1.1. *On an admissible Kähler manifold $M = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}) \rightarrow S$, there exists a polynomial $G_x(z)$ in z such that if $G_x(z)$ is positive on $(-1, 1)$ for some $x \in (0, 1)$, then M admits a conical extremal metric with “sufficiently large” angle in the admissible Kähler class corresponding to x .*

The notations in Theorem 1.1 will be given in Section 2. Theorem 1.1 gives a criterion to determine whether there exist conical admissible Kähler metrics on an admissible manifold. Moreover, following the same arguments in [3] we can show that the existence of conical admissible extremal metrics is equivalent to the positivity of a polynomial. In [18], Tønnesen-Friedman gave an interesting example which admits no extremal metrics in some admissible Kähler classes. However, using the arguments of Theorem 1.1 we can show that it admits conical extremal metrics in any admissible Kähler class.

Corollary 1.2. *On the admissible manifold $\mathbb{P}(\mathcal{O} \oplus \mathcal{L}) \rightarrow \Sigma$ where Σ is a Riemann surface with genus $g(\Sigma) > 1$, there exists a conical extremal metric in any admissible Kähler class.*

Next we will study the relations between conical extremal metrics and energy functionals. Recall that in the smooth case, G. Tian conjectured in [19] that the existence of extremal Kähler metrics is equivalent to the properness of the modified K -energy, which is a generalization of Mabuchi’s K -energy by Guan [14] and Simanca [17]. In [3], using the theory of Chen-Tian [8] a sufficient and necessary condition is given for the existence of general extremal metrics on an admissible manifold. Their results can be extended to conical admissible extremal metrics except the arguments using Chen-Tian’s results. However, if we only consider the *admissible* Kähler metrics, we can show the following result:

Theorem 1.3. *Let $M = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}) \rightarrow S$ be an admissible manifold. The following properties are equivalent for a conical admissible Kähler class Ω :*

- (1) *M admits an extremal Kähler metric in Ω ;*
- (2) *The extremal polynomial $F_\Omega(z)$ is positive on $(-1, 1)$;*
- (3) *The modified K -energy is proper on Ω .*

The admissible manifolds and the extremal polynomials were introduced in [3], and we will explain all the details in Section 2. The equivalence of part (1) and part (2) of Theorem 1.3 is due to [3]. The proof on the properness of the modified K energy relies on Donaldson [11] and Zhou-Zhu's work [22], but we need to carefully study the energy functionals in our situation. In the Kähler-Einstein case, G. Tian prove the equivalence of the existence of Kähler-Einstein metrics and the properness of the energy functionals in [20].

Now we study the modified K -energy on admissible manifolds. The lower boundedness of the modified K -energy is very subtle and it is conjectured by X. X. Chen in [9] [10] that it is equivalent to the property that the infimum of the modified Calabi energy is zero, and it might be related to the existence of extremal metrics with singularities. On the admissible manifolds, we can verify this conjecture and give the full criteria on the modified K -energy in terms of the extremal polynomial:

Theorem 1.4. *Let $M = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}) \rightarrow S$ be an admissible manifold. Then the following properties are equivalent for a conical admissible Kähler class Ω :*

- (1) *The modified K -energy is bounded from below on Ω ;*
- (2) *The extremal polynomial $F_\Omega(z)$ is nonnegative on $(-1, 1)$;*
- (3) *The infimum of the modified Calabi energy on Ω is zero.*

Moreover, if $F_\Omega(z)$ is nonnegative and has m distinct repeated roots z_i on $(-1, 1)$, then M can split into $m + 1$ parts, and each part admits an admissible extremal Kähler metric with generalized cusp singularities at the ends $z = z_i$.

The generalized cusp singularity is defined in Section 5.2, and it is a generalization of the cusp singularity. Combining Theorem 1.4 with Theorem 1.3, we know that the modified K -energy is bounded from below but not proper if and only if the extremal polynomial is nonnegative and has repeated roots on $(-1, 1)$. The phenomena that M may admit complete extremal metrics on each parts is similar to the result of G. Székelyhidi in [16], where he discussed the minimizers of the Calabi energy. It is easy to find an admissible manifold such that the extremal polynomial satisfies this property. For example, we check Tønnesen-Friedman's example as in Corollary 1.2 and have the following:

Corollary 1.5. *On the admissible manifold $M = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}) \rightarrow \Sigma$ where Σ is a Riemann surface with genus $g(\Sigma) > 1$, there is a point $x_s \in (0, 1)$ such that for the admissible Kähler class $\Omega(x, 1)$ with $x \in (0, 1)$,*

- (1) *if $x \in (0, x_s)$, then M admits a smooth admissible extremal metric on $\Omega(x, 1)$;*
- (2) *if $x = x_s$, then the modified K -energy is bounded from below but not proper on $\Omega(x, 1)$. M can split into two parts, and each part admits an admissible extremal metric with a cusp singularity on the fibre;*

- (3) if $x \in (x_s, 1)$, M can split into three parts, two of which has positive extremal polynomials and admit admissible extremal metrics with conical singularities on the fibre, and one has negative extremal polynomial which determines no admissible extremal metrics with singularities.

The above results give close relations between the modified K -energy and the existence of the extremal metrics. In a general admissible manifold, the set of all admissible Kähler classes can be divided into two subsets: one admits extremal metrics and the other doesn't. The boundary Kähler classes of the two subsets have the property that the modified K -energy is bounded from below but not proper. We expect that these properties can be extended to toric manifolds, and we will explore this in a forthcoming paper.

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2 Admissible Kähler metrics

In this section, we recall some basic facts on the admissible Kähler manifolds from [3]. The general admissible Kähler manifolds are defined in [3] and here we only consider a special case for simplicity.

Definition 2.1. A projective vector bundle of the form $M = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}) \rightarrow S$ is called an admissible manifold if M satisfies the following properties.

1. S is a compact complex manifold covered by a product $\tilde{S} = S_1 \times S_2 \times \cdots \times S_N$ of simply connected Kähler manifold (S_i, g_i, ω_i) of complex dimension d_i . Every metric g_i has constant scalar curvature $S_{g_i} = 2d_i s_i$. \mathcal{L} denotes a holomorphic line bundle over S .
2. z is a Morse-Bott function on M with image $[-1, 1]$ and the critical set $z^{-1}(\{-1, 1\})$, and $M^0 := z^{-1}((-1, 1))$ is a principal \mathbb{C}^* bundle over \tilde{S} .
3. There are real numbers $x_i \in (0, 1), i = 1, \dots, N$ such that the metric on M^0 is Kähler:

$$g = \sum_{i=1}^N \frac{1+x_i z}{x_i} g_i + \frac{dz^2}{\Theta(z)} + \Theta(z) \theta^2; \quad (2.1)$$

$$\omega_g = \sum_{i=1}^N \frac{1+x_i z}{x_i} \omega_i + dz \wedge \theta, \quad (2.2)$$

where θ is a connection 1-form with $\theta(K) = 1$ and $d\theta = \sum_i \omega_i$. Here $K = J\nabla_g z$ is a Killing vector field generating the \mathbb{S}^1 action on M . $\Theta(z)$ is a smooth function on $[-1, 1]$ with

$$\Theta(\pm 1) = 0, \quad \Theta(z) > 0, \quad z \in (-1, 1) \quad (2.3)$$

and satisfies some additional conditions which we will describe below.

In [3], the function $\Theta(z)$ satisfies the boundary conditions $\Theta'(\pm 1) = \mp 2$ so that the metric g can extend to M . In the present paper, we allow that each fibre of the admissible manifold M admits conical singularities. Consider the fiber metric

$$g_f = \frac{dz^2}{\Theta(z)} + \Theta(z)\theta^2, \quad (2.4)$$

we define the conical singularities below:

Definition 2.2. A metric g on a Riemann surface Σ is called conical with angle $2\pi\kappa$ of order γ at a point $p \in \Sigma$, if there is a neighborhood U of p such that g can be written in polar coordinates as

$$g = ds^2 + (\kappa^2 s^2 + O(s^{2+\gamma}))\theta^2$$

for some $\kappa, \gamma > 0$.

Note that in the definition 2.2 the metric g is singular at the point p for $\kappa \in (0, 1)$ and g is degenerate at p for $\kappa > 1$. Now we give some boundary conditions on $\Theta(z)$ such that each fibre has conical singularities. For the purpose of simplicity, we assume that each fibre has the singularities with the same angle $2\pi\kappa$ at $z = \pm 1$. Define the set of functions for $\kappa > 0$

$$\mathcal{A}(\kappa) = \{\Theta(z) \in C^\infty[-1, 1] \mid \Theta(z) > 0, z \in (-1, 1), \Theta(\pm 1) = 0, \Theta'(\pm 1) = \mp 2\kappa\}.$$

Note that $\kappa = 1$ is exactly the smooth case discussed in [3].

Lemma 2.3. If $\Theta(z) \in \mathcal{A}(\kappa)$ for some $\kappa > 0$, then the fibre metric g_f defined by (2.4) has conic singularities with angle $2\pi\kappa$ of order 2.

Proof. We only consider the neighborhood near $z = -1$. Define a function $s = s(z)$ by

$$s(z) = \int_{-1}^z \frac{dz}{\sqrt{\Theta(z)}}.$$

Since $\Theta(z) \in \mathcal{A}(\kappa)$, we can check that

$$\frac{d}{ds}\Theta \Big|_{s=0} = \frac{d^3}{ds^3}\Theta \Big|_{s=0} = 0, \quad \frac{d^2}{ds^2}\Theta \Big|_{s=0} = 2\kappa^2, \quad \frac{d^4}{ds^4}\Theta \Big|_{s=0} = 4\kappa^2\Theta''(-1),$$

which implies that

$$g_f = ds^2 + (\kappa^2 s^2 + O(s^4))\theta^2.$$

The lemma is proved. □

The metric of the form (2.1) for some smooth function $\Theta(z) \in \mathcal{A}(\kappa)$ is called a conical admissible Kähler metric with angle $2\pi\kappa$. The complex structure on the fibre will change when the function $\Theta(z)$ varies. However, after a diffeomorphism every Kähler metric defined by different functions $\Theta(z)$ can be viewed as in the same Kähler class, which is called conical admissible Kähler class and denoted by $\Omega(x, \kappa)$.

We can calculate the scalar curvature of an admissible Kähler metric.

Lemma 2.4. (cf. [3]) *The scalar curvature of an admissible metric g is given by*

$$S_g = \sum_{i=1}^N \frac{2d_i s_i x_i}{1 + x_i z} - \frac{F''(z)}{p_c(z)},$$

where $p_c(z) = \prod_{i=1}^N (1 + x_i z)^{d_i}$ and $F(z) = \Theta(z)p_c(z)$.

The advantage of an admissible metric is that its scalar curvature only depends on z . This directly implies that an admissible metric is extremal if and only if the scalar curvature is an affine linear function of z .

Now we look for a function $\Theta(z) \in \mathcal{A}(\kappa)$ such that the corresponding admissible metric g is extremal with the scalar curvature $S_g + Az + B = 0$ for some constants A and B . For any $\Theta(z) \in \mathcal{A}(\kappa)$, the function $F(z) = \Theta(z)p_c(z)$ must satisfy the conditions

$$F(\pm 1) = 0, \quad F'(-1) = 2\kappa p_c(-1), \quad F'(1) = -2\kappa p_c(1), \quad (2.5)$$

and $F(z) > 0$ on $(-1, 1)$. To construct admissible extremal metrics, we define

Definition 2.5. (cf. [3]) *For an admissible Kähler class $\Omega(x, \kappa)$, the extremal polynomial $F_\Omega(z)$ is the function satisfying $F_\Omega(\pm 1) = 0$ and*

$$F''_\Omega(z) = \left(Az + B + \sum_i \frac{2d_i s_i x_i}{1 + x_i z} \right) \cdot p_c(z), \quad z \in (-1, 1). \quad (2.6)$$

Here the constants A and B are given by

$$A\alpha_1 + B\alpha_0 = -2\beta_{0,\kappa}, \quad A\alpha_2 + B\alpha_1 = -2\beta_{1,\kappa}, \quad (2.7)$$

where α_r and $\beta_{r,\kappa}$ are defined by

$$\alpha_r = \int_{-1}^1 p_c(t) t^r dt \quad (2.8)$$

$$\beta_{r,\kappa} = \kappa p_c(1) + (-1)^r \kappa p_c(-1) + \int_{-1}^1 \sum_i \frac{d_i s_i x_i}{1 + x_i t} p_c(t) t^r dt. \quad (2.9)$$

As in Proposition 8 of [3], there is a unique polynomials $F_\Omega(z)$ satisfying the conditions in Definition 2.5. Moreover, by the uniqueness of $F_\Omega(z)$, we have the following existence result:

Theorem 2.6. *On an admissible Kähler manifold M , there is an admissible extremal Kähler metric with angle $2\pi\kappa$ in an admissible Kähler class $\Omega(x, \kappa)$ if and only if $F_\Omega(z)$ is positive on $(-1, 1)$.*

The proof of Theorem 2.6 is the same as in the case $\kappa = 1$ of Proposition 8 in [3] and we omit it here. In fact, using Chen-Tian's results of [8], the result in [3] says that the existence of a general extremal metric in an admissible Kähler class is equivalent to the positivity of the extremal polynomial on $(-1, 1)$. Thus, we would like to ask whether the conical version of Chen-Tian's results hold and whether we can generalize all the results in [3] to the conical case.

3 Existence of conical extremal metrics

In this section, we will show a sufficient condition for the existence of conical admissible extremal metrics, and give an example which admits no smooth extremal metrics in some admissible Kähler classes, but does admit conical extremal metrics in any admissible Kähler classes.

Following the arguments in [3], we have the result:

Theorem 3.1. *On an admissible Kähler manifold M , there exists a polynominal $G_x(z)$ in z which depends only on the function $p_c(z)$ such that if $G_x(z)$ is positive on $(-1, 1)$ for some $x \in (0, 1)$, then M admits a conical extremal metric with "sufficiently large" angle of order 2 in the admissible Kähler class corresponding to x .*

Proof. Here we following the notations in Section 2. It suffices to find when the extremal polynomial $F_\Omega(z)$ is positive for $z \in (-1, 1)$. Note that (2.5) implies

$$\int_{-1}^1 F_\Omega''(z) dz = -2\kappa(p_c(1) + p_c(-1)) \quad (3.1)$$

$$\int_{-1}^1 F_\Omega''(z) z dz = -2\kappa(p_c(1) - p_c(-1)). \quad (3.2)$$

Integrating (2.6) and using (3.1)-(3.2), we have

$$A\alpha_1 + B\alpha_0 = -2\beta_{0,\kappa}, \quad A\alpha_2 + B\alpha_1 = -2\beta_{1,\kappa}, \quad (3.3)$$

where α_r and $\beta_{r,\kappa}$ are defined in Definition 2.5. Direct calculation shows that

$$A = \frac{2(\beta_{0,\kappa}\alpha_1 - \beta_{1,\kappa}\alpha_0)}{\alpha_0\alpha_2 - \alpha_1^2}, \quad B = \frac{2(\alpha_1\beta_{1,\kappa} - \alpha_2\beta_{0,\kappa})}{\alpha_0\alpha_2 - \alpha_1^2}.$$

Note that (2.6) and (3.1) implies that

$$F_\Omega(z) = 2\kappa p(-1)(z+1) + \int_{-1}^z \left(At + B + \sum_{i=1}^N \frac{2d_i s_i x_i}{1+x_i t} \right) p_c(t)(z-t) dt.$$

Observe that $F_\Omega(z)$ is a linear function of κ , and we need the coefficient of κ is positive for $z \in (-1, 1)$. The coefficient of κ in the expression of $\frac{1}{2}(\alpha_0\alpha_2 - \alpha_1^2)F_\Omega(z)$ is

$$\begin{aligned} G_x(z) : &= (\alpha_0\alpha_2 - \alpha_1^2)p(-1)(z+1) \\ &+ \left((\alpha_1 - \alpha_0)p_c(1) + (\alpha_1 + \alpha_0)p_c(-1) \right) \int_{-1}^z p_c(t)(z-t)t dt \\ &+ \left((\alpha_1 - \alpha_2)p_c(1) - (\alpha_1 + \alpha_2)p_c(-1) \right) \int_{-1}^z p_c(t)(z-t) dt, \end{aligned}$$

which depends only on the function $p_c(z)$. Since $\alpha_0\alpha_2 - \alpha_1^2 > 0$, $F_\Omega(z)$ is positive for $z \in (-1, 1)$ if $G_x(z) > 0$ ($z \in (-1, 1)$) and κ is large enough. The theorem is proved. \square

The condition $G_x(z) > 0$ ($z \in (-1, 1)$) is less restrictive than the positivity of the extremal polynomial, and it might be true for any admissible class. Here we discuss the example by C. Tønnesen-Friedman in [18] where we can calculate the angle κ explicitly.

Example: Let Σ be a compact Riemann surface with constant curvature metric $(g_\Sigma, \omega_\Sigma)$, and M be $P(\mathcal{O} \oplus \mathcal{L}) \rightarrow \Sigma$ where \mathcal{L} is a holomorphic line bundle such that $c_1(L) = \frac{1}{2\pi}[\omega_\Sigma]$. Let $2s$ be the scalar curvature of g_Σ . By the Gauss-Bonnet theorem, we have

$$s = \frac{2(1 - g(\Sigma))}{\deg \mathcal{L}},$$

where $g(\Sigma)$ is the genus of Σ . We consider the admissible Kähler metrics of the form

$$g = \frac{1+xz}{x}g_\Sigma + \frac{dz^2}{\Theta(z)} + \Theta(z)\theta^2, \quad x \in (0, 1), \quad (3.4)$$

where $\Theta(z) \in \mathcal{A}(\kappa)$. By [18] and [3], if $s \geq 0$ then there exist extremal metrics in any admissible Kähler classes. However, when $s < 0$ there exist no extremal metrics in some admissible Kähler class. If we allow each fibre has conical singularities, then we have the result:

Theorem 3.2. *On the admissible manifold M with genus $g(\Sigma) > 1$ as above, for any $x \in (0, 1)$ there exists a conical extremal metric with angle $2\pi\kappa$ with*

$$\kappa > \frac{-sx^2}{(1-x)(3+x)}$$

in the admissible Kähler class corresponding to x .

Proof. Here $p_c(z) = 1+xz$. We want to find the extremal polynomial $F_\Omega(z) = \Theta_\Omega(z)(1+xz)$ such that $S_g + Az + B = 0$ for two constants A and B . By Lemma 2.4, we have

$$F_\Omega''(z) = (1+xz) \left(\frac{2sx}{1+xz} + Az + B \right). \quad (3.5)$$

Note that $F_\Omega(z)$ satisfies the boundary conditions

$$F_\Omega(\pm 1) = 0, \quad F_\Omega'(-1) = 2\kappa(1-x), \quad F_\Omega'(1) = -2\kappa(1+x). \quad (3.6)$$

Thus, $F_\Omega''(z)$ satisfies

$$\int_{-1}^1 F_\Omega''(z) dz = -4\kappa, \quad \int_{-1}^1 z F_\Omega''(z) dz = -4\kappa x. \quad (3.7)$$

Combining (3.5)-(3.7), we have

$$A = \frac{6x(sx - 2\kappa)}{3 - x^2}, \quad B = \frac{6(\kappa x^2 - sx - \kappa)}{3 - x^2},$$

and the function $F_\Omega(z)$ can be written as

$$F_\Omega(z) = \frac{(1-z^2)}{2(3-x^2)} \left((2\kappa x^2 - sx^3)z^2 + (6\kappa x - 2\kappa x^3)z + 6\kappa + sx^3 - 4\kappa x^2 \right). \quad (3.8)$$

We want to find when F_Ω is positive for $z \in (-1, 1)$. Let

$$\begin{aligned} Q(z) &= (2\kappa x^2 - sx^3)z^2 + (6\kappa x - 2\kappa x^3)z + 6\kappa + sx^3 - 4\kappa x^2 \\ &= (6xz - 2x^3z + 6 + 2x^2z^2 - 4x^2)\kappa - x^3sz^2 + x^3s. \end{aligned} \quad (3.9)$$

Note that the polynomial $6xz - 2x^3z + 6 + 2x^2z^2 - 4x^2$ is strictly positive for all $x \in (0, 1)$ and $z \in (-1, 1)$, thus M admits a conical extremal metric in any admissible class. It is easy to find a sharper bound of κ . In fact, we can check that

$$Q(1) = (6\kappa - 2\kappa x^2)(1+x) > 0, \quad Q(-1) = (6\kappa - 2\kappa x^2)(1-x) > 0.$$

Thus, $Q(z)$ is positive on $(-1, 1)$ if the following inequality holds

$$-\frac{6\kappa x - 2\kappa x^3}{2(2\kappa x^2 - sx^3)} < -1, \quad \text{or} \quad \kappa > \frac{-sx^2}{(1-x)(3+x)}.$$

The theorem is proved. □

4 Estimates

In this section, we will give some estimates on the modified K -energy and J functional which will be used in the proof of main theorems.

4.1 The symplectic potential

For any $\Theta(z) \in \mathcal{A}(\kappa)$, we define the symplectic potential $u(z)$ of the admissible Kähler metric corresponding $\Theta(z)$ by

$$u''(z) = \frac{1}{\Theta(z)}. \quad (4.1)$$

Note that the symplectic potential is unique up to an affine linear function. Let g_c be the admissible metric with its Kähler form ω_c defined by $\Theta_{c,\kappa}(z) = \kappa(1 - z^2) \in \mathcal{A}(\kappa)$, and we can choose its symplectic potential to be

$$u_{c,\kappa}(z) = \frac{1}{2\kappa} \left((1 - z) \log(1 - z) + (1 + z) \log(1 + z) \right).$$

Denoted by \mathcal{C}_κ the space of functions $u \in C^0([-1, 1])$ satisfying $u''(z) > 0$ on $(-1, 1)$ and

$$u - u_{c,\kappa} \in C^\infty([-1, 1]), \quad u(0) = u'(0) = 0.$$

We can check that for any $u \in \mathcal{C}_\kappa$ the function $\frac{1}{u''}$ belongs to $\mathcal{A}(\kappa)$, and thus it defines a conical admissible metric with angle $2\pi\kappa$.

Now we relate the symplectic potential to the Kähler potential. For any symplectic potential $u \in \mathcal{C}_\kappa$, we define the Legendre transform by

$$y = u'(z), \quad \varphi(y) = -u(z) + yz, \quad (4.2)$$

where $z = (u')^{-1}(y)$ can be viewed as a function of y . We can check that

$$\varphi'_y(y) = z, \quad \varphi''_{yy}(y) = \Theta(z) > 0, \quad z \in (-1, 1).$$

Here we denote $\varphi'_y(y) = d\varphi/dy$ and $\varphi''_{yy}(y) = d^2\varphi/dy^2$ for simplicity. Note that the complex structure defined by (2.1) and (2.2) on the fibre is given by

$$Jdz = \Theta(z)\theta, \quad J\theta = -\frac{1}{\Theta(z)}dz,$$

we have $Jdy = \theta$ and the equalities

$$dJd\varphi = dJ(\varphi'_y(y)dy) = d(z\theta) = z \sum_{i=1}^N \omega_i + dz \wedge \theta = \omega_g - \sum_{i=1}^N \frac{1}{x_i} \omega_i.$$

Now fix an admissible Kähler form ω_c and its complex structure J_c , we have the result:

Lemma 4.1. (cf. [3]) *There exists a fibre-preserving diffeomorphism Ψ on M such that $\Psi^*J = J_c$ and $\Psi^*y = y_c$. Thus, any admissible Kähler metric ω defined by $\Theta(z)$ can be view as in the same Kähler class*

$$\Psi^*\omega = \omega_c + dJ_c d(\varphi(y_c)). \quad (4.3)$$

Thus, the admissible Kähler class is identified with the space \mathcal{C}_κ .

We denote by $\Omega(x, \kappa)$ the admissible Kähler class determined by Lemma 4.1. Any metric in the Kähler class $\Omega(x, \kappa)$ can be written as

$$\omega_g = \sum_{i=1}^N \frac{1}{x_i} \omega_i + dJ_c d\varphi(y_c) = \omega_c + dJ_c d(\varphi(y_c) - \varphi_c(y_c)).$$

4.2 The modified K -energy

The modified K -energy is defined for extremal Kähler metrics by Guan [14] and Simanca [17], and it is a generalization of the K -energy defined by Mabuchi for Kähler-Einstein metrics.

Let g be a Kähler metric on a compact Kähler manifold M , G be a maximal compact connected subgroup of reduced automorphism group and P_g the space of Killing potentials with respect to any G -invariant metric g in the admissible Kähler class Ω . Define pr_g the L^2 -projection to P_g . The modified K -energy is defined by

$$\mu_{g_0}(\varphi) = - \int_0^1 \int_M \frac{\partial \varphi_t}{\partial t} pr_{g_t}^\perp S_{g_t} \omega_{g_t}^n \wedge dt, \quad (4.4)$$

where φ_t is a path in the space of Kähler potentials which connects 0 and φ and $\omega_{g_t} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_t$. It can be shown that the functional $\mu_{g_0}(\varphi)$ is independent of the choice of the path φ_t .

Lemma 4.2. (cf. [3]) *If g is an admissible metric defined by $\Theta(z) \in \mathcal{A}(\kappa)$, then the L^2 projection of S_g orthogonal to the space of Killing potentials is*

$$pr_g^\perp S_g = \frac{F_\Omega''(z) - F''(z)}{p_c(z)},$$

where F_Ω is the extremal polynomial of $\Omega(x, \kappa)$ and $F(z) = \Theta(z)p_c(z)$.

For an admissible Kähler metric in $\Omega(x, \kappa)$, we still define the modified K -energy by (4.4). Note that for an admissible metric, we have the volume form

$$dV_g = p_c(z) \left(\bigwedge_{i=1}^N \frac{1}{d_i! x_i^{d_i}} \omega_i^{d_i} \right) \wedge dz \wedge \theta.$$

Using Lemma 4.2 and integrating by parts, we have

$$\begin{aligned} \mu_{g_c}(\varphi) &= C_1 \cdot \int_0^1 dt \int_{-1}^1 \frac{\partial u}{\partial t} (F_\Omega''(z) - F''(z)) dz \\ &= C_1 \cdot \int_0^1 dt \int_{-1}^1 \frac{\partial u''}{\partial t} (F_\Omega(z) - F(z)) dz \\ &= C_1 \cdot \int_{-1}^1 \left(-p_c(z) \log \frac{u''(z)}{u''_{c,\kappa}(z)} + F_\Omega(z)(u''(z) - u''_{c,\kappa}(z)) \right) dz, \end{aligned} \quad (4.5)$$

where $C_1 = 2\pi \text{Vol}(S, \Pi_i \frac{\omega_i}{x_i})$ and we used the fact that $F(z)$ satisfies the same boundary conditions as $F_\Omega(z)$. Thus, we have the lemma:

Lemma 4.3. *The modified K -energy $\mu_{g_c}(\varphi)$ is a positive multiple of the functional*

$$\mathcal{F}(u) = - \int_{-1}^1 p_c(z) \log \frac{u''(z)}{u''_{c,\kappa}(z)} dz + \int_{-1}^1 F_\Omega(z) (u''(z) - u''_{c,\kappa}(z)) dz,$$

where $u \in \mathcal{C}_\kappa$.

It is proved by Chen-Tian [8] that if a compact Kähler manifold admits an extremal metric, then the modified K -energy is bounded from below. Following the argument in [3], we can easily prove if there is a conical admissible extremal metric in $\Omega(x, \kappa)$, then the modified K -energy is bounded from below in $\Omega(x, \kappa)$. We will improve this result later.

4.3 The J functional

In this section, we follow Zhou-Zhou [22] to discuss when the K -energy is proper. Recall that the J functional defined by Aubin on the space of Kähler potentials,

$$J_g(\varphi) = \frac{1}{V} \int_0^1 \int_M \frac{\partial \varphi_t}{\partial t} (\omega_g^n - \omega_{g_t}^n) \wedge dt \quad (4.6)$$

where φ_t is a path of Kähler potentials connecting 0 to φ . As in the study of Kähler-Einstein metric by Tian [20], we introduce

Definition 4.4. *The K -energy is called proper if there is an increasing function $\rho(t)$ on \mathbb{R} with the property that*

$$\lim_{t \rightarrow +\infty} \rho(t) = +\infty,$$

such that for any Kähler potential φ ,

$$\mu_{\omega_g}(\varphi) \geq \rho(J_{\omega_g}(\varphi)).$$

Recall that any function $u \in \mathcal{C}_\kappa$ can be written as $u = u_{c,\kappa} + v$ for a smooth function v on $[-1, 1]$. The Kähler potential of u and $u_{c,\kappa}$ is related by

$$\varphi(z) = -u(z) + u'(z)z, \quad \varphi_{c,\kappa}(z) = -u_{c,\kappa}(z) + u'_{c,\kappa}(z)z.$$

Thus, the function $\tilde{\varphi} := \varphi - \varphi_{c,\kappa}$ is given by

$$\tilde{\varphi}(z) = -v(z) + v'(z)z, \quad \tilde{\varphi}'_z(z) = v''_{zz}z, \quad \tilde{\varphi} \in C^\infty[-1, 1]. \quad (4.7)$$

To estimate $J_{\omega_{g_c}}(\tilde{\varphi})$ in the admissible Kähler class, we have the result:

Lemma 4.5. *There exists a uniform constant C such that for all $u \in \mathcal{C}_\kappa$ the corresponding $\tilde{\varphi}$ satisfies*

$$\left| J_{\omega_{g_c}}(\tilde{\varphi}) - C_1 \cdot \int_{-1}^1 u(z) dz \right| \leq C, \quad (4.8)$$

where $C_1 = 2\pi \text{Vol}(S, \Pi_i \frac{\omega_i}{x_i})$.

Proof. We follow the argument of Zhou-Zhu [22] to prove the lemma. By the definition (4.6) of $J_{\omega_{g_c}}(\tilde{\varphi})$, we have

$$\begin{aligned} J_{\omega_{g_c}}(\tilde{\varphi}) &= \frac{1}{V} \int_M \tilde{\varphi} \omega_{\omega_{g_c}}^n - \frac{1}{V} \int_0^1 \int_M \frac{\partial \tilde{\varphi}}{\partial t} \omega_{\varphi_t}^n \wedge dt \\ &= \frac{1}{V} \int_M \tilde{\varphi} \omega_{\omega_{g_c}}^n + \frac{2\pi}{V} \text{Vol}(S, \Pi_{i=1}^N \frac{\omega_i}{x_i}) \int_{-1}^1 (u(z) - u_{c,\kappa}(z)) p_c(z) dz. \end{aligned} \quad (4.9)$$

Thus, it suffices to show that $\int_M \tilde{\varphi} \omega_{\omega_{g_c}}^n$ is uniformly bounded from above and below.

Claim 4.6. *We have*

$$\tilde{\varphi}(z) \leq \frac{1}{V} \int_M \tilde{\varphi} \omega_{\omega_{g_c}}^n + C, \quad (4.10)$$

for a uniform constant C .

Note that (4.10) is proved by the Green's function in [22], but we lack the lower bound of Green's function for conical metrics here. However, we can prove it by direct calculation.

Proof of Claim 4.6. In fact, recall the fibre metric of g_c is given by

$$g_{c,f} = \frac{dz^2}{\kappa(1-z^2)} + \kappa(1-z^2)\theta^2.$$

Since $\tilde{\varphi}$ is a Kähler potential and depends only on z , its Laplacian satisfies

$$\Delta_{\omega_{g_c}} \tilde{\varphi} = \left((1-z^2) \tilde{\varphi}'_z \right)'_z \geq -C \quad (4.11)$$

for a constant $C > 0$. Integrating (4.11) from -1 to z and from z to 1 respectively, we have

$$-\frac{C}{1-z} \leq \varphi'_z \leq \frac{C}{1+z}.$$

Fix $z_0 \in [-1, 1]$, for any $z \geq z_0$ we have

$$\tilde{\varphi}(z) - \tilde{\varphi}(z_0) = \int_{z_0}^z \tilde{\varphi}'_z(t) dt \geq - \int_{z_0}^z \frac{C}{1-t} dt = C(\log(1-z) - \log(1-z_0)),$$

and integrating z from z_0 to 1 , we have

$$(1-z_0)\tilde{\varphi}(z_0) \leq \int_{z_0}^1 \tilde{\varphi}(z) dz + C. \quad (4.12)$$

On the other hand, for $z \leq z_0$ we have

$$\tilde{\varphi}(z_0) - \tilde{\varphi}(z) = \int_z^{z_0} \tilde{\varphi}'_z(t) dt \leq \int_z^{z_0} \frac{C}{1+t} dt = C(\log(1+z_0) - \log(1+z)),$$

and integrating from -1 to z_0 we have

$$(z_0 + 1)\tilde{\varphi}(z_0) \leq \int_{-1}^{z_0} \tilde{\varphi}(z) dz + C. \quad (4.13)$$

Combining the inequalities (4.12)-(4.13) we have

$$\tilde{\varphi}(z_0) \leq \frac{1}{2} \int_{-1}^1 \tilde{\varphi}(z) dz + C,$$

and the inequality (4.10) is proved. □

Recall that the functions φ and φ_c defined by u and $u_{c,\kappa}$ respectively satisfy

$$y = u'(z), \quad y_c = u'_{c,\kappa}(z).$$

Thus, $dy/dy_c > 0$ and y can be viewed as a function of y_c for all $y_c \in \mathbb{R}$. For this reason, we still denote by $\varphi = \varphi(y_c)$ as a function of y_c .

Claim 4.7. *We have the inequality*

$$\left| \frac{d\varphi}{dy_c} \right| \leq 1.$$

Proof of Claim 4.7. Since $\tilde{\varphi}$ is a Kähler potential, the function $\varphi = \varphi_c + \tilde{\varphi}$ is convex in y_c . Thus, we have

$$\varphi(y_c) - \varphi(y_0) \geq z_0(y_c - y_0)$$

where $z_0 = \frac{d\varphi}{dy_c}|_{y_c=y_0}$ for any $y_0 \in \mathbb{R}$. Thus, the function $\varphi(y) - z_0 y$ is bounded from below on \mathbb{R} . Direct calculation shows that

$$\varphi_c(y_c) = \log \frac{e^{y_c} + e^{-y_c}}{2},$$

and there is a uniform constant C such that

$$\left| \varphi_c(y_c) - |y_c| \right| \leq C, \quad y_c \in \mathbb{R}.$$

Therefore, for any $y_c \in \mathbb{R}$ we have

$$|y_c| - z_0 y_c \geq \varphi_c(y_c) - z_0 y_c - C \geq \varphi(y_c) - z_0 y_c - C', \quad (4.14)$$

which is bounded from below. Here we used the fact that $\tilde{\varphi} = \varphi(y_c) - \varphi_c(y_c)$ is a bounded function on \mathbb{R} . Since $|y_c| - z_0 y_c$ is a piecewise linear function and bounded from below, we have $z_0 \in [-1, 1]$ and the lemma is proved. □

Define the set

$$\Omega_N = \{\xi \in M \mid \tilde{\varphi}(\xi) \leq \sup_M \tilde{\varphi} - N\}.$$

Thus, we can check that $\text{Vol}_{\omega_{g_c}}(\Omega_N) \rightarrow 0$ as $N \rightarrow +\infty$. In fact, since

$$\begin{aligned} \frac{1}{V} \int_M \tilde{\varphi} \omega_{g_c}^n &= \frac{1}{V} \int_{\Omega_N} \tilde{\varphi} \omega_{g_c}^n + \frac{1}{V} \int_{M \setminus \Omega_N} \tilde{\varphi} \omega_{g_c}^n \\ &\leq \frac{\text{Vol}(\Omega_N)}{V} (\sup_M \tilde{\varphi} - N) + \frac{V - \text{Vol}(\Omega)}{V} \sup \tilde{\varphi} \\ &= \sup_M \tilde{\varphi} - \frac{N \text{Vol}(\Omega_N)}{V}. \end{aligned}$$

Combining this with the inequality (4.10), we have

$$\text{Vol}(\Omega_N) \leq \frac{CV}{N} \rightarrow 0, \quad N \rightarrow +\infty.$$

On the other hand, since $\tilde{\varphi}$ satisfies

$$\tilde{\varphi}(0) = 0, \quad \left| \frac{d\tilde{\varphi}}{dy_c} \right| \leq 1, \quad (4.15)$$

we have

$$\tilde{\varphi}(y_c) = \tilde{\varphi}(y_c) - \tilde{\varphi}(0) \leq \sup_M \left| \frac{d\tilde{\varphi}}{dy_c} \right| \cdot |y| \leq |y|.$$

Thus, for any $y_c \in (-1, 1)$ we have $\tilde{\varphi}(y_c) \leq 1$. Note that $\text{Vol}_g(\{p \in M \mid |y(p)| \leq 1\})$ is strictly positive, but the volume of the set $\Omega_N \rightarrow 0$. Thus, there exists $y_0 \in [-1, 1]$ such that

$$1 \geq \tilde{\varphi}(y_0) \geq \sup_M \tilde{\varphi}(y) - N$$

for N sufficiently large. Thus, $\sup_M \tilde{\varphi}(y) \leq N + 1$ and $\int_M \tilde{\varphi} \omega_{g_c}^n$ is bounded from above. On the other hand, by Claim 4.6 we have

$$\frac{1}{V} \int_M \tilde{\varphi} \omega_{g_c}^n \geq \tilde{\varphi}(0) - C = -C,$$

where we used (4.15). Since $\int_M \tilde{\varphi} \omega_{g_c}^n$ is bounded from above and below, by (4.9) we have the inequality (4.8). Thus, the lemma is proved. \square

Define the operator \mathcal{L} on \mathcal{C}_κ by

$$\mathcal{L}u = \int_{-1}^1 F_\Omega(z) u''(z) dz = \int_{-1}^1 F_\Omega''(z) u(z) dz - F_\Omega'(1) u(1) + F_\Omega'(-1) u(-1).$$

We have the result:

Lemma 4.8. *If there exists a constant $\delta > 0$ such that the inequality*

$$\mathcal{L}u \geq \delta \int_{-1}^1 u(z) dz,$$

holds for any $u \in \mathcal{C}_\kappa$, then there exists a $\lambda > 0$ such that for any $u \in \mathcal{C}_\kappa$ we have

$$\mathcal{F}(u) \geq \lambda \int_{-1}^1 u(z) dz - C_\lambda.$$

Proof. We choose a function $v_0 \in \mathcal{C}_\kappa$ and define a function $G(z)$ by

$$G(z) = \frac{p_c(z)}{v_0''(z)}.$$

Thus, v_0 is a critical point of the functional

$$\begin{aligned} \tilde{\mathcal{F}}(u) &= \int_0^1 dt \int_{-1}^1 \frac{\partial u}{\partial t} (G''(z) - F''(z)) dz \\ &= \int_0^1 dt \int_{-1}^1 \frac{\partial u''}{\partial t} (G(z) - F(z)) dz \\ &= \int_{-1}^1 G(z) (u''(z) - u_{c,\kappa}''(z)) dz - \int_{-1}^1 p_c(z) \log \frac{u''(z)}{u_{c,\kappa}''(z)} dz, \end{aligned}$$

which is a convex functional on \mathcal{C}_κ . Thus, the functional $\tilde{\mathcal{F}}(u)$ is bounded from below,

$$\tilde{\mathcal{F}}(u) \geq \tilde{\mathcal{F}}(v_0) \quad u \in \mathcal{C}_\kappa.$$

For any positive constant $k > 0$ and $u \in \mathcal{C}_\kappa$, we have

$$\begin{aligned} \tilde{\mathcal{F}}\left(\frac{1}{k}u\right) &= \frac{1}{k} \int_{-1}^1 G(z) (u''(z) - k u_{c,\kappa}''(z)) dz - \int_{-1}^1 p_c(z) \log \frac{u''(z)}{k u_{c,\kappa}''(z)} dz \\ &= \frac{1}{k} \tilde{\mathcal{F}}(u) - \frac{k-1}{k} \int_{-1}^1 G(z) u_{c,\kappa}''(z) dz + \log k \int_{-1}^1 p_c(z) dz \\ &= \frac{1}{k} \tilde{\mathcal{F}}(u) - C_k \end{aligned} \tag{4.16}$$

for some constant C_k . Thus, the functional $\tilde{\mathcal{F}}(\frac{1}{k}u)$ is bounded from below on $\tilde{\mathcal{C}}_\kappa$.

Define the functional

$$\tilde{\mathcal{L}}u = \int_{-1}^1 G(z) u''(z) dz = \int_{-1}^1 G''(z) u(z) dz - G'(1)u(1) + G'(-1)u(-1),$$

where we used the fact that $G(z)$ satisfies the same boundary conditions as $F_\Omega(z)$. Note that

$$\begin{aligned} |\mathcal{L}(u) - \tilde{\mathcal{L}}(u)| &= \left| \int_{-1}^1 (G''(z) - F_\Omega''(z))u(z)dz \right| \\ &\leq C \cdot \int_{-1}^1 u(z) dz \leq \frac{C + \delta}{\delta} \cdot \mathcal{L}(u) - \delta \int_{-1}^1 u(z) dz, \end{aligned}$$

where C is a positive constant independent of $u(z)$. Thus, we get

$$\mathcal{L}(u) \geq \frac{\delta}{C + 2\delta} \tilde{\mathcal{L}}(u) + \frac{\delta^2}{C + 2\delta} \int_{-1}^1 u(z) dz$$

and

$$\begin{aligned} \mathcal{F}(u) &\geq \tilde{\mathcal{F}}\left(\frac{\delta}{C + 2\delta} u\right) + \frac{\delta^2}{C + 2\delta} \int_{-1}^1 u(z) dz - \log \frac{\delta}{C + 2\delta} \\ &\geq \frac{\delta^2}{C + 2\delta} \int_{-1}^1 u(z) dz - \log \frac{\delta}{C + 2\delta} - C', \end{aligned}$$

where we used (4.16) in the last inequality. The lemma is proved. \square

5 Proof of main results

In this section, we will use the estimates in Section 4 to prove Theorem 1.3, Theorem 1.4 and Corollary 1.5.

5.1 Proof of Theorem 1.3

In this subsection, we will prove Theorem 1.3. By Theorem 2.6, it suffices to show that

Theorem 5.1. *On an admissible manifold M , there exists an extremal Kähler metric on $\Omega(x, \kappa)$ if and only if the modified K -energy is proper.*

Proof. Suppose that M admits an admissible extremal Kähler metric on $\Omega(x, \kappa)$. To prove the properness of the K -energy, by Lemma 4.5 and Lemma 4.8 it suffices to show that there is a $\delta > 0$ such that for any $u \in \mathcal{C}_\kappa$,

$$\mathcal{L}(u) \geq \delta \int_{-1}^1 u(z) dz. \quad (5.1)$$

In fact, since $F_\Omega(z)$ is positive on $(-1, 1)$ and satisfies the boundary condition (2.5), there is a constant $c > 0$ such that for any $z \in [0, 1]$ we have $F_\Omega(z) \geq c(1 - z)$. Note that u is convex, we have

$$\int_0^1 F_\Omega(z) u''(z) dz \geq c \int_0^1 (1 - z) u''(z) dz = cu(1).$$

Similarly, since $F'_\Omega(-1) > 0$ and $F(z) \geq c'(1+z)(z \in [-1, 0])$ for some constant $c' > 0$, we have the inequality

$$\int_{-1}^0 F_\Omega(z)u''(z) dz \geq c' \int_{-1}^0 (1+z)u''(z) dz = c'u(-1)$$

Combining the above inequalities and taking $\delta = \min\{c, c'\}$, we have

$$\int_{-1}^1 F_\Omega(z)u''(z) dz \geq \delta(u(1) + u(-1)) \geq 2\delta \int_{-1}^1 u dz,$$

where we used the convexity of u in the last inequality. Thus, (5.1) is proved and by Lemma 4.5-4.8 the modified K -energy is proper.

Now we show the necessity part of the theorem. Suppose that the modified K -energy is proper. By Theorem 2.6, we only need to show that the extremal polynomial $F_\Omega(z)$ is positive on $(-1, 1)$. Fix any $u \in \mathcal{C}_\kappa$. For any smooth nonnegative convex function $f(z)$ on $[-1, 1]$ with $f(0) = 0$ and $f'(0) = 0$, the functions $u_k = u + kf \in \mathcal{C}_\kappa$ for any $k \in \mathbb{N}$. We calculate the modified K -energy of u_k ,

$$\begin{aligned} \mathcal{F}(u_k) &= - \int_{-1}^1 p_c(z) \log \frac{u_k''(z)}{u_{c,\kappa}''(z)} dz + \mathcal{L}(u_k) - \int_{-1}^1 F_\Omega(z)u_{c,\kappa}''(z) dz \\ &\leq - \int_{-1}^1 p_c(z) \log \frac{u''(z)}{u_{c,\kappa}''(z)} dz + \mathcal{L}(u) + k\mathcal{L}(f) - \int_{-1}^1 F_\Omega(z)u_{c,\kappa}''(z) dz \\ &= \mathcal{F}(u) + k\mathcal{L}(f). \end{aligned} \tag{5.2}$$

Using the inequality (5.2), we have

Claim 5.2. *If the modified K -energy is bounded from below, then the extremal polynomial $F_\Omega(z)$ is nonnegative on $(-1, 1)$.*

Proof. The claim is due to [3] and we give the details here for completeness. Suppose that $F_\Omega(z)$ is negative at some point on $(-1, 1)$. Then we can choose a nonnegative smooth function $r(z)$ on $(-1, 1)$ such that

$$\int_{-1}^1 F_\Omega(z)r(z)dz < 0.$$

Let u_k be a sequence of functions in \mathcal{C}_κ satisfying $u_k''(z) = u_{c,\kappa}''(z) + kr(z)$. As $k \rightarrow +\infty$ we have

$$\mathcal{F}(u_k) \leq \mathcal{F}(u_{c,\kappa}) + k \int_{-1}^1 F_\Omega(z)r(z)dz \rightarrow -\infty,$$

where we used (5.2). Thus, the K -energy is not bounded from below, a contradiction. □

Using Claim 5.2, we can construct a sequence of functions with some special properties.

Claim 5.3. *If $F_\Omega(z)$ is nonnegative but not positive on $(-1, 1)$, then there is a sequence of smooth convex functions $f_k(z)$ ($k \in \mathbb{N}$) on $[-1, 1]$ with $f_k(0) = 0$, $f'_k(0) = 0$ and*

$$\lim_{k \rightarrow +\infty} \mathcal{L}(kf_k) = 0, \quad \lim_{k \rightarrow +\infty} \int_{-1}^1 kf_k(z) dz = +\infty. \quad (5.3)$$

Proof. Define the function $\eta(s)$ on by

$$\eta(s) = \begin{cases} e^{\frac{1}{s^2-1}}, & |s| \leq 1; \\ 0, & |s| > 1. \end{cases} \quad (5.4)$$

which is a smooth function on \mathbb{R} . Let $h_k(s) = k \cdot \eta(k(s - z_0))$ and define

$$f_k(z) = \int_0^z (z - s) h_k(s) ds.$$

Then we can check that $f_k(z)$ is a smooth convex function on $(-1, 1)$ and satisfies

$$f_k(0) = 0, \quad f'_k(0) = 0, \quad f''_k(z) = h_k(z).$$

Note that for any $z_0 \in (-1, 1)$ we have

$$\lim_{k \rightarrow +\infty} \int_{-1}^1 f_k(z) dz = \frac{1}{2} \int_{-1}^1 (1 - |z_0|) e^{\frac{1}{t^2-1}} dt > 0.$$

Thus, we have

$$\lim_{k \rightarrow +\infty} \int_{-1}^1 kf_k(z) dz = +\infty.$$

By Claim 5.2 the extremal polynomial $F_\Omega(z)$ is nonnegative on $(-1, 1)$. If $F_\Omega(z)$ is not positive on $(-1, 1)$, it has repeated roots on $(-1, 1)$. Near a root $z_0 \in (-1, 1)$, $F_\Omega(z)$ can be expressed as $F_\Omega(z) = g(z)(z - z_0)^{2m}$ for a positive function $g(z)$ on $[z_0 - \epsilon, z_0 + \epsilon]$ and an integer $m \geq 1$. Thus, we have

$$\begin{aligned} \mathcal{L}(kf_k) &= \int_{-1}^1 kF_\Omega(z) f''_k(z) dz = \int_{-1}^1 kF_\Omega(z) h_k(z) dz \\ &= \int_{-1}^1 kF_\Omega\left(z_0 + \frac{t}{k}\right) e^{\frac{1}{t^2-1}} dt \\ &= k^{1-2m} \int_{-1}^1 g\left(z_0 + \frac{t}{k}\right) t^{2m} e^{\frac{1}{t^2-1}} dt \rightarrow 0, \end{aligned}$$

as $k \rightarrow +\infty$. The claim is proved. □

Now we proceed to prove the necessity part of the theorem. Since $\mathcal{F}(u)$ is proper, there is an increasing function $\rho(t)$ such that $\lim_{t \rightarrow +\infty} \rho(t) = +\infty$ and

$$\mathcal{F}(u) \geq \rho\left(\int_{-1}^1 u(z) dz\right), \quad \forall u \in \mathcal{C}_\kappa. \quad (5.5)$$

If F_Ω is not positive on $(-1, 1)$, by Claim 5.3 we can construct $u_k = u + kf_k$ with the property (5.3). Combining this with the inequality (5.5) and (5.2), we have

$$\rho\left(\int_{-1}^1 (u + kf_k)(z) dz\right) \leq \mathcal{F}(u) + \mathcal{L}(kf_k) \rightarrow \mathcal{F}(u),$$

as $k \rightarrow +\infty$. However, the left hand side will tend to infinity, which is a contradiction. Thus, $F_\Omega(z)$ is positive on $(-1, 1)$ and the theorem is proved. \square

5.2 Proof of Theorem 1.4

In this section, we will prove Theorem 1.4. Theorem 1.4 follows from the following two results:

Theorem 5.4. *On an admissible manifold M , the K -energy is bounded from below if and only if $F_\Omega(z)$ is nonnegative on $\Omega(x, \kappa)$.*

Proof. The necessity part is proved in Claim 5.2. We only need to show the sufficiency part. Assume that F_Ω is nonnegative on $(-1, 1)$. If it is positive, then by Theorem 1.3 the K -energy is proper. Thus, it suffices to consider the case when $F_\Omega(z)$ has repeated roots on $(-1, 1)$. By the expression of the K -energy,

$$\mathcal{F}(u) = \int_{-1}^1 \left(-p_c(z) \log u''(z) + F(z)u''(z) \right) dz + C.$$

Note that for any $a > 0$, we have the inequality,

$$ax - \log x \geq 1 + \log a, \quad x \in (0, \infty).$$

Thus, for any convex function u , we have

$$-p_c(z) \log u''(z) + F_\Omega(z)u''(z) \geq p_c(z)\left(1 + \log \frac{F_\Omega(z)}{p_c(z)}\right).$$

Since $p_c(z)$ is positive on $[-1, 1]$, we only need to check whether the integral

$$\int_{-1}^1 \log F_\Omega(z) dz > -\infty. \quad (5.6)$$

In fact, near a root $z_0 \in (-1, 1)$, the polynomial can be expressed as $F_\Omega(z) = g(z)(z - z_0)^{2m}$ for some smooth function $g(z)$ which is positive on $[z_0 - \epsilon, z_0 + \epsilon]$ and $m \in \mathbb{N}$. Here $\epsilon > 0$ is sufficiently small such that $F_\Omega(z)$ has no other roots. Note that

$$\int_{z_0 - \epsilon}^{z_0 + \epsilon} \log F_\Omega(z) dz = \int_{z_0 - \epsilon}^{z_0 + \epsilon} \log g(z) dz + m \int_{z_0 - \epsilon}^{z_0 + \epsilon} \log(z - z_0)^2 dz > -\infty.$$

Thus, the inequality (5.6) holds. The theorem is proved. □

Recall that the modified Calabi energy can be expressed by

$$\int_M (pr^\perp S_g)^2 dV_g = C \int_{-1}^1 \frac{(F''(z) - F_\Omega''(z))^2}{p_c(z)} dz$$

where we used Lemma 4.2. For simplicity, we define the modified Calabi energy by

$$Ca(u) = \int_{-1}^1 \frac{(q''(z))^2}{p_c(z)} dz, \quad q(z) = F(z) - F_\Omega(z).$$

Now we have the result:

Theorem 5.5. *On an admissible manifold M , the infimum of the modified Calabi energy on $\Omega(x, \kappa)$ is zero if and only if $F_\Omega(z)$ is nonnegative on $(-1, 1)$.*

Proof. Suppose that there is an interval $[a, b] \subset (-1, 1)$ such that $F_\Omega(z) \leq -\epsilon$ is negative on $[a, b]$ for some $\epsilon > 0$. Since for any $u \in \mathcal{C}_\kappa$ the function $F(z) = \frac{p_c(z)}{u''(z)}$ is always positive on $(-1, 1)$, we have

$$q(z) = F(z) - F_\Omega(z) \geq \epsilon > 0, \quad z \in [a, b]. \quad (5.7)$$

Note that $F(z)$ satisfies the same boundary conditions (2.6) as $F_\Omega(z)$, the function $q(z)$ has

$$q(\pm 1) = 0, \quad q'(\pm 1) = 0.$$

Therefore, we have the inequality

$$|q(z)| \leq \left| \int_z^1 q''(s)(s - z) ds \right| \leq \frac{1}{\sqrt{3}} \left(\int_z^1 q''^2(s) ds \right)^{\frac{1}{2}} (1 - z)^{\frac{3}{2}}.$$

Combining this inequality with (5.7), we have

$$Ca(u) \geq \frac{1}{\lambda} \int_{-1}^1 q''^2(s) ds \geq \frac{3\epsilon^2}{\lambda(1 - a)^3},$$

where $\lambda = \max\{p_c(z) | z \in [-1, 1]\}$. Thus, the modified Calabi energy has a positive lower bound.

Now we show the sufficiency part of the theorem. If $F_\Omega(z)$ is positive on $(-1, 1)$, then by Theorem 1.3 M admits extremal metrics on $\Omega(x, \kappa)$ and hence the infimum of the modified Calabi energy is zero. If $F_\Omega(z)$ is nonnegative and has repeated roots on $(-1, 1)$, we can choose a sequence of smooth positive functions $F_n(z)$ with the boundary conditions (2.6) such that $F_n(z)$ converges smoothly to $F_\Omega(z)$ on $(-1, 1)$. Then we can show that the modified Calabi energy determined by $F_n(z)$ tends to zero. For example, suppose that $F_\Omega(z)$ has the only root $z_0 \in (-1, 1)$ on $(-1, 1)$. Then we can choose a sequence of functions,

$$F_n(z) = F_\Omega(z) + \frac{1}{n^2} \eta(n(z - z_0)),$$

where η is defined by (5.4), which is positive on $(-1, 1)$ and satisfies the same boundary as $F_\Omega(z)$. Let $u_n(z) \in \mathcal{C}_\kappa$ be the function determined by $F_n(z) = \frac{p_c(z)}{u_n''(z)}$, we have

$$\begin{aligned} Ca(u_n) &= \int_{-1}^1 \frac{1}{p_c(z)} (F_n''(z) - F_\Omega''(z))^2 dz \\ &= \frac{1}{n} \int_{-1}^1 \frac{1}{p_c(z_0 + \frac{t}{n})} \frac{4(3t^4 - 1)^2}{(t^2 - 1)^8} e^{\frac{2}{t^2 - 1}} dt \rightarrow 0. \end{aligned}$$

Similarly, we can show the infimum of the modified Calabi energy is zero if $F_\Omega(z)$ has many repeated roots on $(-1, 1)$. Thus, the theorem is proved. \square

Now we show the last part of Theorem 1.4. First we introduce the singularities of a metric:

Definition 5.6. Let Σ be a Riemann surface with a metric g . A point $p \in \Sigma$ is called

1. a cusp point, if near the point p the metric g can be written as

$$g = \rho_1(s) ds^2 + \rho_2(s) e^{-2s} \theta^2, \quad s \in [s_0, \infty),$$

where $\rho_1(s)$ and $\rho_2(s)$ are positive smooth functions at p .

2. a generalized cusp point, if there is a integer $k \in \mathbb{N}$ such that near p the metric g can be written as

$$g = \rho_1(s) \frac{ds^2}{s^k} + \rho_2(s) s^k \theta^2, \quad s \in (0, s_0],$$

where $\rho_1(s)$ and $\rho_2(s)$ are positive smooth functions at p . In particular, if $k = 2$ we can show that p is a cusp point.

Suppose that $F_\Omega(z)$ is nonnegative and has distinct repeated roots z_i ($1 \leq i \leq m$) with $-1 < z_1 < \dots < z_m < 1$. Let $z_0 = -1$ and $z_{m+1} = 1$. By Part 2 of Definition 2.1, the manifold $M_i = z^{-1}((z_i, z_{i+1}))$ for $0 \leq i \leq m$ is a principal \mathbb{C}^* bundle over \tilde{S} , and $F_\Omega(z)$ is positive on M_i . Thus, M_i admits an admissible extremal Kähler metric and we need to check the behavior of the metric near the ends $z = z_i, z_{i+1}$ if $z_i, z_{i+1} \neq \pm 1$:

Lemma 5.7. *The admissible extremal metric on M_i has generalized cusp singularities at the ends $z = z_i, z_{i+1}$ if $z_i, z_{i+1} \neq \pm 1$.*

Proof. Consider the fibre metric near $z = z_i, z_{i+1}$ ($0 \leq i \leq m$),

$$g_f = \frac{dz^2}{\Theta_\Omega(z)} + \Theta_\Omega(z)\theta^2, \quad z \in (z_i, z_{i+1}).$$

Since z_i is a repeated root of $F_\Omega(z)$, we can write $F_\Omega(z) = g(z)(z - z_i)^{2N}$ ($N \in \mathbb{N}$) where $g(z)$ is positive on $[z_i - \epsilon, z_i + \epsilon]$. Thus, the fibre metric can be written as

$$g_f = \frac{p_c(z)}{g(z)} \left(\frac{dz^2}{(z - z_i)^{2N}} + \frac{g(z)^2}{p_c(z)^2} (z - z_i)^{2N} \theta^2 \right), \quad z \in (z_i, z_i + \epsilon). \quad (5.8)$$

In the special case of $N = 1$, the fibre metric (5.8) has cusp singularities at $z = z_i$. In fact, let $z - z_i = e^{-s}$ and we have

$$g_f = \rho_1(s)ds^2 + \rho_2(s)e^{-2s}\theta^2,$$

which is a metric with a cusp singularity. Here $\rho_1(s)$ and $\rho_2(s)$ are smooth positive functions near $z = z_i$. For general $N \in \mathbb{N}$, the metric (5.8) has a generalized cusp singularity, which is complete near z_i . □

Remark 5.8. *If $F_\Omega(z)$ is negative at some points in $(-1, 1)$, we define*

$$I_+ = \{z \in (-1, 1) \mid F_\Omega(z) > 0\}, \quad I_- = \{z \in (-1, 1) \mid F_\Omega(z) \leq 0\},$$

and $M_+ = z^{-1}(I_+)$. Then M_+ has an admissible extremal metric with singularities.

In fact, near a boundary point of M_+ with $z = z_0 \in \bar{I}_+ \cap \bar{I}_-$ and $(z_0, z_0 + \epsilon) \subset I_+$ for small $\epsilon > 0$, the extremal polynomial can be written as $F_\Omega(z) = g(z)(z - z_0)^{2N-1}$ ($N \in \mathbb{N}$) where $g(z)$ is smooth and positive on $(z_0 - \epsilon, z_0 + \epsilon)$. We can discuss the singularities at $z = z_0$ as in Lemma 5.7. For $N = 1$, the fibre metric at $z = z_0$ has a conical singularity with angle

$$2\pi\kappa = \frac{\pi F'_\Omega(z_0)}{p_c(z_0)} > 0.$$

For $N \geq 2$, the fibre metric has generalized cusp singularities.

5.3 Proof of Corollary 1.5

In this section, we will show that after carefully choosing the parameters, the extremal polynomial of the example in Section 3 is nonnegative and has a repeated root on $(-1, 1)$, hence the modified K -energy is bounded from below but not proper.

For simplicity, we only consider the smooth case $\kappa = 1$ of the example in Section 3. By the equality (3.8) and (3.9), we need to find the parameters x and s such that

$$Q(z) = (2x^2 - sx^3)z^2 + (6x - 2x^3)z + 6 + sx^3 - 4x^2$$

has repeated roots on $(-1, 1)$. Note that since $s < 0$ the minimum point of $Q(z)$

$$-\frac{6x - 2x^3}{2(2x^2 - sx^3)} \in (-1, 1). \quad (5.9)$$

We would like to whether there is a root of

$$\Delta(x) = (6x - 2x^3)^2 - 4(2x^2 - sx^3)(6 + sx^3 - 4x^2) = 0.$$

In fact, we have

Lemma 5.9. *There is a point $x_s \in (-1, 1)$ such that $\Delta(x) < 0$ for $x \in (0, x_s)$ and $\Delta(x) > 0$ for $x \in (x_s, 1)$.*

Proof. By direct calculation, we have

$$\Delta^{(4)}(x) = 192 + 1440x^2 - 2880sx + 1440s^2x^2 > 0, \quad x \in (0, 1).$$

Since $\Delta^{(3)}(0) = 144s < 0$ and $\Delta^{(3)}(1) = 672 - 1296s + 480s^2 > 0$, there is a point $x_3 \in (0, 1)$ such that $\Delta^{(3)}(x) < 0$ for $x \in (0, x_3)$ and $\Delta^{(3)}(x) > 0$ for $x \in (x_3, 1)$. Similarly, since $\Delta''(0) = -24 < 0$ and $\Delta''(1) = 192 - 336s + 120s^2 > 0$, there is a point $x_2 \in (0, 1)$ such that $\Delta''(x) < 0$ for $x \in (0, x_2)$ and $\Delta''(x) > 0$ for $x \in (x_2, 1)$. Now direct calculation show that

$$\Delta'(0) = 0, \quad \Delta'(1) = 32 - 48s + 24s^2 > 0.$$

We can also show that there is a point $x_1 \in (0, 1)$ such that $\Delta'(x) < 0$ for $x \in (0, x_1)$ and $\Delta'(x) > 0$ for $x \in (x_1, 1)$. Combining this with $\Delta(0) = 0, \Delta(1) = 4 + s^2 > 0$, we know there is a point $x_s \in (0, 1)$ such that $\Delta(x) < 0$ for $x \in (0, x_s)$ and $\Delta(x) > 0$ for $x \in (x_s, 1)$. \square

Thus, for any $s < 0$, there is a $x_s \in (0, 1)$ such that $\Delta(x_s) = 0$ and (5.9) holds. For the admissible Kähler class $\Omega(x_s, 1)$, the extremal polynomial $F_\Omega(z)$ is nonnegative and has a repeated root $z_s \in (-1, 1)$. By Theorem 1.3 and Theorem 1.4, the modified K -energy is bounded from below but not proper. Moreover, M can split into two parts $M_0 = z^{-1}((-1, z_s))$ and $M_1 = z^{-1}((z_s, 1))$, and each part admits admissible extremal metrics with a cusp singularity on the fibre.

If $x \in (0, x_s)$, $\Delta(x) < 0$ and $Q(z)$ has no root on $(-1, 1)$. Thus, M admits a smooth admissible extremal metric on $\Omega(x, 1)$. If $x \in (x_s, 1)$, $\Delta(x) > 0$. Note that

$$Q(1) = 6 + 6x - 2x^2 - 2x^3 > 0, \quad Q(-1) = (1 - x)(6 - 2x^2) > 0.$$

Combing this with (5.9), $Q(z)$ has two simple zeros on $(-1, 1)$. Thus, M can split into three parts, two of which satisfy $F_\Omega(z) > 0$ and admit admissible extremal metrics with conical singularities on the fibre by Remark 5.8.

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